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AN INTERPOLATION FORMULA FOR "EQUIDISTANT" FREQUENCY DISTRIBUTIONS.

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A frequency distribution is conveniently represented by the area under some continuous curve. Moreover, the total frequency is usually included between two distinct values of the independent variable, outside of which there is no frequency. Hence, if x denote the independent variable, and $y=f(x)$ denote the curve bounding the distribution, we must have

$$f(a)=f(b)=0, \tag{1}$$

where a and b are the extreme values of x.

The frequency between any two values of x, α and β, is given by $\int_\alpha^\beta y\,dx$. In the usual case the values of this integral are known for each of the sub-intervals obtained by dividing the entire internal $[a, b]$ into equal parts. Let m be the number of sub-intervals, and for convenience choose each sub-interval of unit length. Then

$$b=a+m. \tag{2}$$

Let α_k, where $k=1, 2, 3, \cdots, m$, be the known frequencies over each of the sub-intervals $[a, a+1]$, $[a+1, a+2]$, $\cdots$, $[a+k-1, a+k]$, $\cdots$, $[a+m-1, a+m]$. Then

$$\alpha_k=\int_{a+k-1}^{a+k} y\,dx;\ k=1, 2, 3, \cdots, m. \tag{3}$$

The problem then is to interpolate frequencies for other than the equidistant values of the independent variable. This consists in finding some function $f(x)$ which will satisfy the $m+2$ conditions of (1) and (3).* Since this is infinitely

* Cf. the problem first solved by Lagrange, Oeuvres 1, p. 87. Also articles in Enzyk. d. Math. Wissen. IIA, 9a, p. 647.

indeterminate, the problem reduces to finding a form for $f(x)$ which can be handled practically, and which can be applied in all cases.

For this purpose we may choose for $y=f(x)$ the function

$$y=\sum_{n=1}^{n=m}\frac{n\pi}{m}\cdot a_n\cdot sin\frac{(x-a)\,n\pi}{m}. \tag{4}$$

By (2), this evidently satisfies the conditions (1). The conditions (3) then become

$$a_k=\sum_{n=1}^{n=m}a_n\cdot\left[\cos\frac{(k-1)\,n\pi}{m}-\cos\frac{kn\pi}{m}\right];\ k=1,2,3,\ \cdots,\ m. \tag{5}$$

The m conditions (5) are just sufficient to determine the m constants a_n. The problem now is to evaluate these coefficients.

Suppose the m equations of (5) to be written in succession and that we add the first k of them together, for all values of k. Then, if we put

$$s_k=a_1+a_2+a_3+\cdots+a_k, \tag{6}$$

$$s_k=\sum_{n=1}^{n=m}a_n\cdot\left[1-\cos\frac{n\pi}{m}k\right]=\sum_{n=1}^{n=m}a_n-\sum_{n=1}^{n=m}a_n\cdot\cos\frac{n\pi}{m}k.$$

Letting

$$A=\sum_{n=1}^{n=m}a_n, \tag{7}$$

the last may be written

$$s_k=A-\sum_{n=1}^{n=m}a_n\cos\frac{n\pi}{m}k;\ k=1,2,3,\ \cdots,m. \tag{8}$$

The m conditions of (8) replace those of (5).

Let r be any integer from 1 to m. Then

$$r\leqq m;\ n\leqq m. \tag{9}$$

Now consider the m quantities

$$2\cos\frac{r\pi}{m}k;\ k=1,2,3,\ \cdots,m. \tag{10}$$

Multiplying each of the equations of (8) by the corresponding quantity of (10) and adding the resulting equations we obtain

$$2\sum_{k=1}^{k=m}s_k\cos\frac{r\pi}{m}k=A\cdot I\,(r)+\sum_{n=1}^{n=m}a_n\cdot K\,(n,r), \tag{11}$$

where

(12) $$I(r)=2\sum_{k=1}^{k=m}\cos\frac{r\pi}{m}k,$$

and

(13) $$K(n,r)=-\sum_{k=1}^{k=m}2\cos\frac{n\pi}{m}k\cdot\cos\frac{r\pi}{m}k.$$

From trigonometry, we have

(14) $$\sum_{p=1}^{p=q}\cos p\theta=-\frac{1}{2}+\frac{\sin\left(q\theta+\frac{\theta}{2}\right)}{2\sin\frac{\theta}{2}}.$$

Putting $q=m$, and $\theta=\frac{r\pi}{m}$ in (14), (12) becomes

(15) $$I(r)=-1+\frac{\sin\left[r\pi+\frac{r\pi}{2m}\right]}{\sin\frac{r\pi}{2m}}.$$

By (9), the denominator of (15) never vanishes. Hence:

(16) *if r is even, $I(r)=0$; if r is odd, $I(r)=-2$.*

From trigonometry,

$$2\cos x\cdot\cos y=\cos(x+y)+\cos(x-y).$$

Applying this, (13) becomes

(17) $$K(n,r)=-\sum_{k=1}^{k=m}\left[\cos\frac{(n+r)\pi}{m}k+\cos\frac{(n-r)\pi}{m}k\right]$$
$$=-\sum_{k=1}^{k=m}\cos\frac{(n+r)\pi}{m}k-\sum_{k=1}^{k=m}\cos\frac{(n-r)\pi}{m}k.$$

Again using (14), (17) becomes

(18) $$K(n,r)=\frac{1}{2}-\frac{\sin\left[(n+r)\pi+\frac{(n+r)\pi}{2m}\right]}{2\sin\frac{(n+r)\pi}{2m}}$$
$$+\frac{1}{2}-\frac{\sin\left[(n-r)\pi+\frac{(n-r)\pi}{2m}\right]}{2\sin\frac{(n-r)\pi}{m}}$$

Now suppose $n+r$ is *even*; then $n-r$ is even, and the angles

or $\dfrac{n-r}{2m}$ is integral. By (9), $\dfrac{n+r}{2m}$ is integral only when $n=r=m$; $\dfrac{n-r}{2m}$ is integral only when $n=r$. Hence we have:

$$\text{(19)} \qquad \textit{if } n+r \textit{ is even, and } n \neq r,\ K(n_1r)=0.$$

Now suppose $n+r$ is even and $n=r$. In this case (18) is indeterminate. However, on returning to (17) we obtain

$$\text{(18a)} \qquad K(r,r)=\frac{1}{2}-\frac{\sin\left[2r\pi+\dfrac{r\pi}{m}\right]}{2\sin\dfrac{r\pi}{m}}-m=\frac{1}{2}-\frac{\sin\dfrac{r\pi}{m}}{2\sin\dfrac{r\pi}{m}}-m.$$

Hence $K=-m$, unless the denominator is zero. This occurs only when $r=m$. Hence we have:

$$\text{(20)} \qquad \textit{if } r \neq m,\ K(r,r)=-m.$$

Suppose $n=r=m$. Substituting in (17) we get at once:

$$\text{(21)} \qquad K(m,m)=-2m.$$

Consider $n+r$ *odd*. Then $n-r$ is odd, and the angles $(n+r)\pi$ and $(n-r)\pi$ in the brackets of (18) may each be replaced by π. It is clear, furthermore, that in this case the denominators of (18) cannot vanish. Hence we have

$$\text{(22)} \qquad \textit{if } n+r \textit{ is odd, } K(n,r)=2.$$

For definiteness, we shall now suppose m *odd*. Then we may conveniently represent the results of (16), (19), (20), (21), (22), in the following table:

$$r \text{ even} \left\{ \begin{array}{l} n \text{ even} \left\{ \begin{array}{l} n=r \\ \\ n \neq r \end{array} \right. \\ \\ n \text{ odd} \end{array} \right. \qquad \left. \begin{array}{l} K=-m \\ \\ K=\ \ 0 \\ \\ K=\ \ 2 \end{array} \right\} I=0$$

$$r \text{ odd} \begin{cases} n \text{ odd} \begin{cases} n=r \begin{cases} r=m & K=-2m \\ r\neq m & K=-\ m \end{cases} \\ n\neq r & K=\ \ 0 \end{cases} \\ n \text{ even} & K=\ \ 2 \end{cases} \Bigg\} I=-\mathbf{2}.$$

We note that for each value of r from 1 to m we get a set of multipliers in (10), and a new equation (11) from equations (8). We thus obtain m new equations to replace those of (8). That the new set is equivalent to (8) is shown by the fact that a definite solution is obtained.

We may put

$$\text{(23)} \qquad \begin{aligned} M &= a_1+a_3+a_5+\cdots+a_m\,; \\ N &= a_2+a_4+a_6+\cdots+a_{m-1}. \end{aligned}$$

Then

$$A=M+N.$$

Consider first the irregular case $r=m$. From the table we obtain: $I=-2$; for n even, $K=2$; for n odd and $n\neq m$, $K=0$; for $n=r=m$, $K=-2m$. Hence equation (11) takes the form

$$\begin{aligned} 2\sum_{k=1}^{k=m} s_k \cos \pi k &= -2A-2ma_m+2(a_2+a_4+a_6+\cdots+a_{m-1}) \\ &= -2A+2N-2ma_m, \end{aligned}$$

or

$$\text{(24)} \qquad \sum_{k=1}^{k=m} (-1)^k s_k = -M-ma_m.$$

Let r assume any of the *even* values from 2 to $m-1$. From the table, $I=0$; $K=2$ if n is odd; $K=0$ if n is even and $n\neq r$; $K=-m$ if $n=r$. Hence (11) becomes

$$\begin{aligned} 2\sum_{k=1}^{k=m} s_k\cdot \cos \frac{r\pi}{m}k &= 2(a_1+a_3+\cdots+a_m)-ma_r \\ \text{(25)} \qquad &= 2M-ma_r;\ r=2,\,4,\,6,\,\cdots,\,m- \end{aligned}$$

Let r assume any of the *odd* values allowed by (9), exclusive of $r=m$, already considered. From the table we then obtain, $I=-2$; $K=2$ if n is even; $K=0$ if n is odd and $n\neq r$; $K=-m$ if $n=r$. Hence (11) becomes

$$(26)\quad 2\sum_{k=1}^{k=m} s_k\cdot\cos\frac{r\pi}{m}k = -2A+2(a_2+a_4+\cdots+a_{m-1})-ma_r$$
$$= -2M-ma_r;\ r=1, 3, 5, \cdots, m-2.$$

We have then m equations in (24), (25), (26). We now proceed to determine M. Adding together the $\frac{m-1}{2}$ equations of (26), we obtain

$$2\sum_{t=1}^{t=\frac{m-1}{2}}\sum_{k=1}^{k=m} s_k\cdot\cos\frac{(2t-1)\pi k}{m} = -(m-1)M-m(a_1+a_3+a_5+\cdots+a_{m-2})$$

$$(27)\qquad = -(m-1)M-m(M-a_m)$$
$$= -(2m-1)M+ma_m.$$

Adding (24) and (27), we obtain

$$(28)\quad M=-\frac{1}{2m}\left\{\sum_{k=1}^{k=m}(-1)^k s_k+2\sum_{t=1}^{t=\frac{m-1}{2}}\sum_{k=1}^{k=m} s_k\cdot\cos\frac{(2t-1)\pi}{m}k\right\}.$$

This gives us M in terms of definitely known quantities. From (24), (25), (26), the coefficients a_n are then determined. We can, however, decidedly simplify the expression for M in (28).

From trigonometry,

$$\sum_{u=1}^{u=t}\cos(2u-1)\theta=\frac{\sin 2t\theta}{2\sin\theta},$$

unless $\sin\theta=0$. Using this, (28) becomes in order

$$M=-\frac{1}{2m}\left\{\sum_{k=1}^{k=m}(-1)^k s_k+2\sum_{k=1}^{k=m} s_k\sum_{t=1}^{t=\frac{m-1}{2}}\cos(2t-1)\frac{\pi k}{m}\right\}$$

$$=-\frac{1}{2m}\left\{\sum_{k=1}^{k=m}(-1)^k s_k+2\sum_{k=1}^{k=m-1} s_k\sum_{t=1}^{t=\frac{m-1}{2}}\cos(2t-1)\frac{\pi k}{m}\right.$$
$$\left.+2s_m\sum_{t=1}^{t=\frac{m-1}{2}}\cos(2t-1)\pi\right\}$$

$$= -\frac{1}{2m}\left\{\sum_{k=1}^{k=m}(-1)^k s_k + 2\sum_{k=1}^{k=m-1} s_k \frac{\sin\frac{(m-1)\pi k}{m}}{2\sin\frac{\pi k}{m}} - (m-1)s_m\right\}$$

$$= -\frac{1}{2m}\left\{\sum_{k=1}^{k=m}(-1)^k s_k + \sum_{k=1}^{k=m-1} s_k \frac{\sin\pi k\cdot\cos\frac{\pi k}{m} - \cos\pi k\cdot\sin\frac{\pi k}{m}}{\sin\frac{\pi k}{m}} - (m-1)s_m\right\}$$

$$= -\frac{1}{2m}\left\{\sum_{k=1}^{k=m}(-1)^k s_k - \sum_{k=1}^{k=m-1}(-1)^k s_k - (m-1)s_m\right\}$$

$$= -\frac{1}{2m}\left\{-s_m - ms_m + s_m\right\}$$

$$(29)\qquad = \frac{s_m}{2}.^*$$

From (24), (25), (26), we now obtain the coefficients a_n in the following forms:

$$(30)\qquad \left.\begin{aligned} a_m &= -\frac{1}{m}\left\{\frac{s_m}{2} + \sum_{k=1}^{k=m}(-1)^k s_k\right\};\\ a_r &= \frac{2}{m}\left\{\frac{s_m}{2} - \sum_{k=1}^{k=m} s_k\cos\frac{r\pi}{m}k\right\},\ r=2,4,6,\cdots,m-1;\\ a_r &= -\frac{2}{m}\left\{\frac{s_m}{2} + \sum_{k=1}^{k=m} s_k\cos\frac{r\pi}{m}k\right\},\ r=1,3,5,\cdots,m-2.\end{aligned}\right\}$$

If in (4) a_m be replaced by $\frac{a_m}{2}$, the three forms (30) for the coefficients a_r can be expressed in the single form

$$(31)\qquad a_r = (-1)^r\frac{s_m}{m} - \frac{2}{m}\sum_{k=1}^{k=m} s_k\cos\frac{r\pi}{m}k.$$

Suppose now m is *even.* Then we obtain in manner parallel to the case m odd:

$$(28')\qquad M = -\frac{1}{m}\sum_{t=1}^{t=\frac{m}{2}}\sum_{k=1}^{k=m} s_k\cos\frac{(2t-1)\pi}{m}k$$

$$(29')\qquad = \frac{s_m}{2}.$$

* In practical computation, the two values found for M serve as an invaluable check.

The coefficients a_n then become

$$(30') \quad \left.\begin{aligned} a_m &= \frac{1}{m}\left\{\frac{s_m}{2} - \sum_{k=1}^{k=m}(-1)^k s_k\right\}; \\ a_r &= \frac{2}{m}\left\{\frac{s_m}{2} - \sum_{k=1}^{k=m} s_k \cos\frac{r\pi}{m}k\right\}, \; r=2, 4, 6, \cdots, m-2; \\ a_r &= -\frac{2}{m}\left\{\frac{s_m}{2} + \sum_{k=1}^{k=m} s_k \cos\frac{r\pi}{m}k\right\}, \; r=1, 3, 5, \cdots, m-1. \end{aligned}\right\}$$

From (30′) it is clear that (31) gives the coefficients a_n also when m is even. Having expressed the coefficients a_n in terms of the definitely known quantities s_k, the function (4) is then completely determined.*

In practical computation, the labor involved consists primarily in computing the values of $\sum_{k=1}^{k=m} s_k \cos\frac{r\pi}{m}k$ for all values of $r \leqq m$. This requires the m^2 values of $s_k \cos\frac{r\pi}{m}k$. But it is readily observed that all the m^2 values of $\cos\frac{r\pi}{m}k$ are given by only $\frac{m-1}{2}$ of them (if m is odd). Furthermore, s_k is a "constant multiplier" for each of the values of $\cos\frac{r\pi}{m}k$, $r \leqq m$. In practice, further use can be made of the simple relations between the trigonometric functions to materially shorten the work. The usual case arising for interpolation, is to obtain the frequencies for midway divisions of the sub-intervals. The computation made necessary for this is about equivalent to that in obtaining the coefficients a_n from the given frequencies a_k.

*If the number of sub-intervals of $[a, b]$, m, be indefinitely increased, it may be shown without difficulty, as is to be expected, that the function (4) becomes a Fourier series.